RATIONAL BOUNDING IN THE PROBLEM OF THE PLANE STRESS STATE OF AN IDEAL FIBER COMPOSITE

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Problems seeking the optimal anisotropy and rational bonding structure of composite material components in the plane stress or plane deformation state were studied in [1-5]. A criterion for maximal strength of components from laminar anisotropic composites is set up in [1, 2]. Optimality conditions are obtained in [3, 4] and in the case of axial symmetry optimization problems for anisotropic plate stiffness are solved. Optimality conditions are determined in [5] in the problem of maximizing the carrying capacity of a body from an anisotropic plastic material. Using these conditions, optimal bonding schemes [4] can be found numerically. As regards the analytic solutions, they are obtained successfully only for high symmetry of the structure since because of nonlinearity of the optimality conditions the corresponding problem also turns out to be nonlinear. Substantial simplifications can be achieved because of idealization of the rheological model of the material.

A composite medium bonded by two families of inextensible fibers is examined in this paper. Such a medium is called an ideal fibrous composite [6, 7]. The problem of seeking the material bonding scheme is sought for which the loads are perceived just by the fibers while there are no stresses in the binder. Existence conditions are obtained for equally stressed bonding schemes, and it is shown that finding them upon satisfaction of certain conditions reduces to solving a Dirichlet problem for the Laplace equation.

1. EQUATIONS OF THE THEORY OF AN IDEAL FIBROUS COMPOSITE

The model of an ideal fibrous composite is the simplest model of a material comprised of high-modulus fibers and low-modulus matrix and consists of assuming two hypotheses: the fibers are distributed continuously in the matrix and they are inextensible.

We obtain the equilibrium equation of a plane medium bonded by two families of inextensible fibers under the assumption that the stresses in the matrix are zero. Let us consider a plate of unit thickness under plane stress state conditions. In a two-dimension domain Ω that the plate occupies, we introduce a system of curvilinear orthogonal coordinates x_1, x_2 (Fig. 1), and its directions $\mathbf{i}_1$ and $\mathbf{i}_2$. We give the stacking directions of the two fiber families at each point by the unit vectors $\mathbf{j}_1$ and $\mathbf{j}_2$. Let the first fiber make the angle φ with the axis $\mathbf{i}_1$ and the second the angle ψ, i.e., $\mathbf{i}_1\cdot\mathbf{j}_1 = \cos\varphi$, $\mathbf{i}_1\cdot\mathbf{j}_2 = \cos\psi$ (the dot denotes the scalar product of vectors). Let us consider the angles φ and ψ to vary from point to point and be sufficiently smooth functions of the coordinates. The stresses in the material are characterized by the tensor

$$\boldsymbol{\sigma} = \Sigma^{\alpha\beta}\mathbf{j}_\alpha \otimes \mathbf{j}_\beta = \boldsymbol{\sigma}^{\alpha\beta}\mathbf{i}_\alpha \otimes \mathbf{i}_\beta, \tag{1.1}$$

where $\sigma^{\alpha\beta}$ are components of the tensor $\boldsymbol{\sigma}$ in the x_1, x_2 coordinate system, $\Sigma^{\alpha\beta}$ are components of $\boldsymbol{\sigma}$ in the oblique-angled coordinate system associated with the vectors $\mathbf{j}_1$ and $\mathbf{j}_2$, the symbol $\otimes$ denotes the direct product of tensors [8]. Here and henceforth summation from 1 to 2 is performed over pairs of repeated Greek subscripts. Since only the fibers perceive the load in the case under consideration, the matrix of the components $\Sigma^{\alpha\beta}$ of the tensor $\boldsymbol{\sigma}$ has the diagonal form

$$\Sigma^{11} = p,\ \Sigma^{22} = q,\ \Sigma^{12} = \Sigma^{21} = 0.$$

Let us find the explicit form of the matrix $\sigma^{\alpha\beta}$ of the components of the stress tensor $\boldsymbol{\sigma}$ in the x_1x_2 coordinate system. To do this we take into account that $\mathbf{j}_1$, $\mathbf{j}_2$ are obtained from $\mathbf{i}_1$, $\mathbf{i}_2$ by rotation through angles φ and ψ, respectively:

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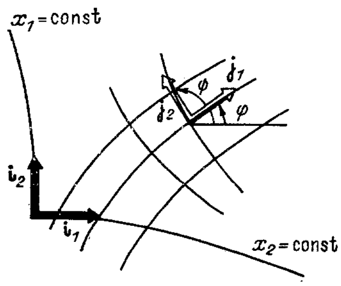

Fig. 1

$$\mathbf{j}_1 = \mathbf{i}_1 \cos\varphi + \mathbf{i}_2 \sin\varphi,\ \mathbf{j}_2 = \mathbf{i}_1 \cos\psi + \mathbf{i}_2 \sin\psi. \tag{1.2}$$

Then from (1.1)

$$\boldsymbol{\sigma} = \Sigma^{\alpha\beta}\mathbf{j}_\alpha \otimes \mathbf{j}_\beta = p\mathbf{j}_1 \otimes \mathbf{j}_1 + q\mathbf{j}_2 \otimes \mathbf{j}_2 = p(\cos^2\varphi\mathbf{i}_1 \otimes \mathbf{i}_1 + 2\sin\varphi\cos\varphi\mathbf{i}_1 \otimes$$

$$\otimes\, \mathbf{i}_2 + \sin^2\varphi\mathbf{i}_2 \otimes \mathbf{i}_2) + q(\cos^2\psi\mathbf{i}_1 \otimes \mathbf{i}_1 + 2\cos\psi\sin\psi\mathbf{i}_1 \otimes \mathbf{i}_2 + \sin^2\psi\mathbf{i}_2 \otimes \mathbf{i}_2) = \sigma^{\alpha\beta}\mathbf{i}_\alpha \otimes \mathbf{i}_\beta,$$

from which

$$\sigma^{\alpha\beta} = \begin{pmatrix} p\cos^2\varphi + q\cos^2\psi & (1/2)\,p\sin 2\varphi + (1/2)\,q\sin 2\psi \\ (1/2)\,p\sin 2\varphi + (1/2)q\sin 2\psi & p\sin^2\varphi + q\sin^2\psi \end{pmatrix}. \tag{1.3}$$

The equilibrium equations of a medium in the case of no bulk forces have the form

$$\begin{aligned} &\frac{\partial}{\partial x_1}(H_2\sigma^{11}) + \frac{\partial}{\partial x_2}(H_1\sigma^{12}) - \sigma^{22}\frac{\partial H_2}{\partial x_1} + \sigma^{12}\frac{\partial H_1}{\partial x_2} = 0, \\ &\frac{\partial}{\partial x_1}(H_2\sigma^{12}) + \frac{\partial}{\partial x_2}(H_1\sigma^{22}) - \sigma^{11}\frac{\partial H_1}{\partial x_2} + \sigma^{12}\frac{\partial H_2}{\partial x_1} = 0 \end{aligned} \tag{1.4}$$

(H_1, H_2 are Lame coefficients of the coordinate system x_1x_2). Substitution of (1.3) into (1.4) yields

$$\begin{aligned} &H_2\frac{\partial}{\partial x_1}(p\cos^2\varphi + q\cos^2\psi) + \frac{H_1}{2}\frac{\partial}{\partial x_2}(p\sin 2\varphi + q\sin 2\psi) + \\ &+ (p\sin 2\varphi + q\sin 2\psi)\frac{\partial H_1}{\partial x_2} + (p\cos 2\varphi + q\cos 2\psi)\frac{\partial H_2}{\partial x_1} = 0, \\ &\frac{H_2}{2}\frac{\partial}{\partial x_1}(p\sin 2\varphi + q\sin 2\psi) + H_1\frac{\partial}{\partial x_2}(p\sin^2\varphi + q\sin^2\psi) - \\ &- (p\cos 2\varphi + q\cos 2\psi)\frac{\partial H_1}{\partial x_2} + (p\sin 2\varphi + q\sin 2\psi)\frac{\partial H_2}{\partial x_1} = 0. \end{aligned} \tag{1.5}$$

Let us consider that static boundary conditions

$$\sigma^{11}n_1 + \sigma^{12}n_2 = T^1,\ \sigma^{12}n_1 + \sigma^{22}n_2 = T^2, \tag{1.6}$$

are given everywhere on the domain boundary, where T^1, T^2 are surface force vector components, n_1, n_2 are projections of the unit normal to the boundary on the $\mathbf{i}_1, \mathbf{i}_2$ axes. The boundary conditions in terms of p and q take the form

$$\begin{aligned} &(p\cos^2\varphi + q\cos^2\psi)n_1 + (1/2)(p\sin 2\varphi + q\sin 2\psi)n_2 = T^1, \\ &(1/2)(p\sin 2\varphi + q\sin 2\psi)n_1 + (p\sin^2\varphi + q\sin^2\psi)n_2 = T^2. \end{aligned} \tag{1.7}$$

Let us note that for given φ and ψ the boundary value problem (1.5) and (1.7) to determine the functions p and q is not always solvable. This means that the fiber system characterized by the stacking angles φ and ψ and loaded by the forces T^1 and T^2 is statically nonequilibrated. In a composite medium bonded by inextensible elements, this nonequilibration is compensated because of stress in the matrix. Reasoning in reverse order, it is easy to arrive at the deduction that the solvability of (1.5) and (1.7) indicates the absence of stress in the matrix of the composite medium under consideration that is loaded in conformity with (1.6).

Before going over to formulation of the problem of rational bonding, we mention the physical meaning of p and q. The fiber tension vectors **F** and **G** have the following components ($\mathbf{F} = F_x\mathbf{i}_1 + F_y\mathbf{i}_2$, $\mathbf{G} = G_x\mathbf{i}_1 + G_y\mathbf{i}_2$): in a Cartesian coordinate system

$$F_x = f\cos\varphi,\ F_y = f\sin\varphi,\ G_x = g\cos\psi,\ G_y = g\sin\psi$$

(f and g are the fiber tensions of the first and second families). Let us now introduce the fiber concentrations η_1 and η_2 by defining them as the number of fibers of each family passing through a single section perpendicular to the fiber direction here. Evaluating the projection of the total force on a segment of unit length perpendicular to the fiber direction, we obtain $p = f\eta_1$, $q = g\eta_2$. Thus, p and q are the fiber tensions referred to unit length.

2. EQUALLY STRONG BONDING SCHEME BY TWO NONORTHOGONAL FIBER SYSTEMS

The equations (1.5) and (1.7) were considered in Sec. 1 as relationships to determine the fiber tensions p and q. Let us formulate the inverse problem. Let it now be required to find the distribution of fiber stacking angles in the x_1x_2 plane given by the functions $\varphi(x_1, x_2)$, $\psi(x_1, x_2)$ so that the stress in the fibers would satisfy the equalities

$$p(x_1,\ x_2) = P,\ q(x_1,\ x_2) = Q, \tag{2.1}$$

where P, Q are given quantities. For an equally strong bonding scheme P and Q are identical constants.

It is expedient to start construction of an equally strong bonding scheme by seeking the solution of the boundary value problem for σ^{11}, σ^{22}, σ^{12} in which the relationships (1.4) are supplemented by the equation

$$\sigma^{11} + \sigma^{22} = P + Q = \text{const} \tag{2.2}$$

and the boundary conditions

$$\begin{gathered}\sigma^{11}(s) = T^1n_1 - T^2n_2 + n^2{}_2(P+Q),\\ \sigma^{22}(s) = T^2n_2 - T^1n_1 + n^2{}_1(P+Q),\quad \sigma^{12}(s) = T^1n_2 + T^2n_1 - n_1n_2(P+Q),\end{gathered} \tag{2.3}$$

obtained from (1.4) with (2.2) taken into account.

In the general case the boundary value problem (1.4), (2.2), and (2.3) is overdefined, however, if the functions $T^1(s)$, $T^2(s)$ satisfy a certain additional relationship, it turns out to be correct. Let us clarify the above for the case when the x_1x_2 coordinate system agrees with a global xy Cartesian coordinate system. Taking (2.2) into account, (1.4) take on the form

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0,\quad \frac{\partial\sigma_{xy}}{\partial x} - \frac{\partial\sigma_{xx}}{\partial y} = 0. \tag{2.4}$$

The relationships (2.4) are Cauchy–Riemann conditions and, therefore, σ_{xx} and σ_{xy} are conjugate harmonic functions

$$\Delta\sigma_{xx} = 0,\ \Delta\sigma_{xy} = 0 \tag{2.5}$$

(Δ is the Laplace operator). It is easy to note that the pair σ_{yy} and σ_{xy} also possess the property mentioned.

The presence of the two boundary conditions (2.3) for two harmonically conjugate functions overdefines the problem since the correct formulation consists in giving just one of the conditions (2.3) on the boundary of the domain Ω. However if the functions $T_x(s)$ and $T_y(s)$ are subjected to a definite relationship, the boundary value problem (2.3) and (2.4), and additionally the problem (1.4), (2.2), and (2.3) are normally solvable. The derivation of this relationship is the following. The Dirichlet problem is solved for (2.5), say, with the first condition (2.3) and the harmonic function $\sigma_{xx}(x, y)$ is determined. Later the con-

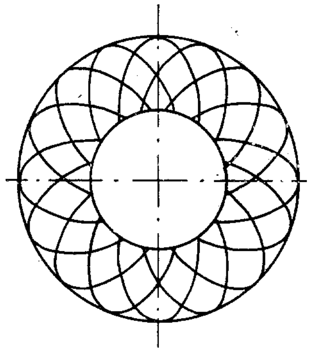

Fig. 2

jugate harmonic function $\sigma_{xy}(x, y)$ is sought by integrating (2.3). The boundary values $\sigma_{xx}(s)$ and $\sigma_{xy}(s)$ are thereby determined. The force components $T_x(s)$ and $T_y(s)$ should satisfy the system of equalities (2.3) identically along the whole boundary. If this condition cannot be satisfied at some point of the boundary, then this means that the equally strong solution for the given boundary conditions $T_x(s)$ and $T_y(s)$ in the whole domain including the boundary is not found successfully.

We now assume that the forces $T^1(s)$ and $T^2(s)$ are such that the stresses $\sigma^{11}(x_1, x_2)$, $\sigma^{22}(x_1, x_2)$, $\sigma^{12}(x_1, x_2)$ are determined in the domain Ω as functions of the parameter $P + Q$. Finding the bonding angles reduces to solving a transcendental system of algebraic equations for $\varphi(x_1, x_2)$, $\psi(x_1, x_2)$:

$$P\cos 2\varphi + Q\cos 2\psi = \sigma^{11} - \sigma^{22},\ P\sin 2\varphi + Q\sin 2\psi = 2\sigma^{12}. \tag{2.6}$$

Let us obtain the solution of this system of equations. Transferring terms containing ψ to the right side, squaring both equations and combining, we have $2Q(a\cos 2\psi + b\sin 2\psi) = a^2 + b^2 + Q^2 - P^2 (a = \sigma^{11} - \sigma^{22},\ b = 2\sigma^{12})$. This equation and an analogous one for φ reduce to

$$\begin{aligned} \sin 2(\alpha + \psi) &= (R^2 + Q^2 - P^2)/2QR, \\ \sin 2(\alpha + \varphi) &= (R^2 + P^2 - Q^2)/2PR. \end{aligned} \tag{2.7}$$

Here $R(P + Q)$ is the second invariant of the stress tensor deviator $R(P + Q) = \sqrt{(\sigma^{11} - \sigma^{22})^2 + 4(\sigma^{12})^2}$. The angle α is defined by the equality $\tan 2\alpha = a/b$. Let us note that not all the roots of (2.7) satisfy (2.6) although the converse assertion is, it is understood, valid and it is necessary for the solvability of (2.6) and (2.7) that P and Q correspond to definite constraints. The domain of allowable P and Q is given by the system of inequalities

$$\begin{aligned} \max_{(x_1,x_2)\in\Omega} \frac{|R^2 + Q^2 - P^2|}{|2QR|} &\leqslant 1, \\ \max_{(x_1,x_2)\in\Omega} \frac{|R^2 - Q^2 + P^2|}{|2PR|} &\leqslant 1. \end{aligned} \tag{2.8}$$

If the pair of values of P and Q selected initially does not satisfy the constraints (2.8), a material with other values of the fiber strength should be chosen.

3. RATIONAL BONDING OF A DISC LOADING UNIFORMLY AT THE EDGES

To illustrate the method proposed for seeking the rational plate bonding scheme, we present an example of computing an equally strong bonding scheme for a disc loaded along the inner radius. Let the boundary conditions be given in the form $(r_1 < r_2)$

$$\sigma_r(r_1) = -T,\ \sigma_r(r_2) = 0,\ \sigma_{\theta r}(r_1) = \sigma_{\theta r}(r_2) = 0. \tag{3.1}$$

Let us seek the bonding scheme satisfying the relationship

$$P = Q = (1/2)K,\ \psi(r) = -\varphi(r). \tag{3.2}$$

The equalities $\sigma_{\theta r}(\theta, r) \equiv 0$, $\alpha = \pi/4$ are a corollary of (3.2). The relationships (1.4) and (2.2) are written in the form

$$\frac{d}{dr}[\sigma_r r] = \sigma_\theta, \quad \sigma_r + \sigma_\theta = K. \tag{3.3}$$

The solution of the boundary value problem (3.1) and (3.3) exists for values of the parameters T and K connected by the relationships $K = 2Tr_1{}^2/(r_2{}^2 - r_1{}^2)$ and has the form

$$\sigma_r = \frac{Tr_1^2}{r_2^2 - r_1^2}(1 - r_2^2/r^2), \quad \sigma_\theta = \frac{Tr_1^2}{r_2^2 - r_1^2}(1 + r_2^2/r^2).$$

The bonding angle is given by the expression

$$\varphi = (1/2)\arccos(-r_2{}^2/r^2). \tag{3.4}$$

The fiber stacking lines corresponding to (3.4) are indicated in Fig. 2.

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